International Journal of Theoretical Physics, Vol. 39, No. 3, 2000

Restricting and Extending States and Positive Maps on Operator Algebras[†]

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Received December 8, 1999

The aim of this paper is to summarize, deepen, and comment upon some recent results concerning restrictions and extensions of states on operator algebras. The first part is focused on the question of the circumstances under which a pure state or a completely positive map restricts to a pure state on maximal Abelian subalgebra. In the second part we present an extension theorem for Stone-algebra-valued measures on quotionts of JBW algebras and discuss its consequences.

The goal of this note is to summarize recent results concerning relationship between positive maps on operator algebras and their maximal Abelian subalgebras. More specifically, in the first part we concentrate on the question of when a pure state (resp. pure positive map) restricts to a pure, i.e., multiplicative, state on a maximal abelian Subalgebra—a maximal classical subsystem in physical interpretetion. The second part is devoted to the extension for Stone-algebra-valued measures on projections of JBW algebras. These themes have the physical significance of relating C^*-algebra quantum mechanics to classical physics and stem naturally from the structure theory of operator algebras.

Let us recall a few notions and fix the notation. Throughout the paper A will denote a JB algebra (unital or nonunital), i.e., a real Banach algebra whose product $\circ$ satisfies algebraic properties

(a) $x \circ y = y \circ x$
(b) $x \circ (y \circ x^2) = (x \circ y) \circ x^2$

and the norm of which obeys C^*-algebraic conditions:

[†] This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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0020-7748/00/0300-0721$18.00/0

(1) $\|x^2\| = \|x\|^2$
(2) $\|x^2\| \leq \|x^2 + y^2\|$

(For additional information on JB algebras we refer to ref. 8). The theory of JB algebras generalizes many aspects of the theory of C^*-algebras. Indeed, any self-adjoint part of a C^*-algebra equipped with the product $x \circ y = 1/2(xy + yx)$ is a JB algebra. Associative JB algebras are exactly self-adjoint parts of Abelian C^*-algebras. (This fact is far from true for general JB algebras.) Therefore Abelian C^*-algebras become associative algebras in the Jordan context.

Finally, let us recall that a linear map $T: A \to B$ between two JB algebras is positive if $T(a^2) \geq 0$ for all $a \in A$ (order is given by the cone of all positive elements). A *state* on A is a norm-one positive functional on A. A pure state is an extreme point of the convex set of states on A. In the sequel we denote by $B(H)$ and $B(H)_{sa}$ the algebra of all bounded operators on a Hilbert space H and its self-adjoint part, respectively.

1. RESTRICTION PROPERTIES OF PURE STATES AND MAPS

Let ρ be a pure state on a C^*-algebra $\mathcal{A}$. By Kadison's transitivity theorem [e.g., 12], ρ is a norm attaining functional, i.e., $\rho(a) = \|\rho\| = 1$ for some self-adjoint $a \in \mathcal{A}$. (This fact is trivial if $\mathcal{A}$ is unital.) Using the Schwarz inequality, we can deduced that

$$\rho(ab) = \rho(a)\rho(b) \qquad \text{for any} \quad b \in \mathcal{A}$$

In particular, ρ is multiplicative, i.e., pure, on a singly generated algebra $C^*(a)$. A natural question arises whether $C^*(a)$ can be enlarged to a maximal Abelian subalgebra $\mathcal{B}$ such that ρ is still pure on $\mathcal{B}$. Though this problem has not been solved in general, it was proved in a surprisingly wide range of circumstances, including most physically meaningful ones, that pure state on a C^*-algebra does restrict to a $*$-homomorphism on some maximal Abelian subalgebra. In particular, it was proved by Aarnes and Kadison that any pure state on a separable unital C^*-algebra restricts to a multiplicative state on some maximal Abelian subalgebra [1]. Akemann sharpened this result to the following form: Given pure states $\rho_1, \rho_2, \ldots, \rho_n$ on a separable (unital or nonunital) C^*-algebra that are mutually orthogonal (i.e., $\|\rho_i - \rho_j\| = 2$ for $i \neq j$), we can always find a maximal Abelian subalgebra such that all states $\rho_1, \rho_2, \ldots, \rho_n$ are multiplicative on it [2]. In the case of individual states, the result of Akemann was generalized by Barnes, who showed that any pure state on a C^*-algebra with GNS representation on a separable Hilbert space is multiplicative on some maximal Abelian subalgebra [4]. In further developments the restriction property of pure states was proved for pure states on type I C^*-algebras and normal pure states on von Neumann algebras [4, 5, 16]. As opposed to C^*-algebras, which are adopted for largely pragmatic reasons, JB algebras are more natural for quantum theory because all physically relevant structures (projection lattices, state spaces, positive operators) are in fact Jordan structures. So it is both mathematically and physically interesting to extend results on pure states on C^*-algebras to the more general context of JB algebras. Our main result along this line is the following theorem [11]:

Theorem 1. Let $\rho_1, \rho_2, \ldots, \rho_n$ be pairwise orthogonal pure states on a JB algebra A such that their central supports $c(\rho_i)A^{**}$ are σ-finite for all $i = 1, \ldots, n$. Then there is a maximal associative subalgebra B of A such that each ρ_i ($i = 1, \ldots, n$) is a multiplicative (i.e., pure) state on B.

The central support $c(\rho)$ of a given state ρ is defined as a smallest central projection in A^{**} such that canonical normal extension $\tilde{\rho}$ of ρ to A^{**} is concentrated at it, i.e., $c(\rho) = \wedge\{p \in A^{**} | p$ is a central projection and $\tilde{\rho}(p) = 1\}$. If A is a self-adjoint part of a C^*-algebra and ρ is a pure state, then $c(\rho)A^{**} \simeq B(H_\rho)_{sa}$, where H_ρ is a Hilbert space resulting from the GNS representation of ρ. In that case $c(\rho)$ is a σ-finite projection exactly when H_ρ is separable. Thus, Theorem 1 is a proper generalization of the Akermann-Barnes results even in the framework of C^*-algebras. In the case of JB algebras new types of irreducible representations (spin-factor, quaternionic types) occur. For that reason the generalization of results on the restriction property of pure states on C^*-algebras for JB algebras is not straightforward and requires new ideas (including, e.g., a Jordan version of the transitivity theorem, etc.).

As an application of Theorem 1, we shall consider the restriction property of completely positive maps on C^*-algebras. Let us assume that $\mathscr{A}$ is a unital C^*-algebra and H a Hilbert space. Let $M_n(\mathscr{A})$ denote a C^*-algebra of all $n \times n$ matrices over $\mathscr{A}$. A mapping $\varphi: \mathscr{A} \to B(H)$ is called *completely positive* if $\varphi^{(n)}$ is positive for all n, where $\varphi^{(n)}: M_n(\mathscr{A}) \to M_n(B(H))$ is defined by

$$\varphi^{(n)}(a_{ij})_{ij} = (\varphi(a_{ij}))_{ij}$$

According to the Stinespring theorem, any unital completely positive map φ is similar to a *-representation $\pi: \mathscr{A} \to B(H_\pi)$ in the sense of the decomposition

$$\varphi = v^*\pi v$$

where $v: H \to H_\pi$ is an isometry and $[\pi v(H_\pi)] = H_\pi$. (Here $[X]$ denotes the closed linear span of a set X.) Let us remark that any completely positive map is positive, and the inverse of this statement holds provided that either the domain or the range of the map is Abelian. In particular, any state is completely positive and the set $S_H(\mathscr{A})$ of all completely positive unital maps of $\mathscr{A}$ to $B(H)$ can be viewed as a generalized state space (in physics, quantization of the state space). The operator-valued state spaces $S_H(\mathscr{A})$ are being intensively studied at the present time for their importance for operator algebras and their applications in quantum theory. There are a few possibilities in the convexity theory accompanying complete positive maps for how to define extreme points. We say that $\varphi \in S_H(\mathscr{A})$ is *pure* if it spans an extreme ray, i.e., if $\psi \leq \psi$, $\varphi \in S_H(\mathscr{A})$, implies $\psi = \lambda\varphi$ for some $\lambda \in [0, 1]$. (We write $\psi \leq \varphi$ if $\varphi - \psi$ is completely positive.) Further, $\varphi \in S_H(\mathscr{A})$ is called a (linear) *extreme point* if it is an extreme point of the convex set $S_H(\mathscr{A})$. Finally, the most important notion in the operator-valued convexity theory is the concept of C^*-extreme point. A map $\varphi \in S_H(\mathscr{A})$ is a C^*-extreme point if the following implication holds: If $\varphi = \sum_{i=1}^n t_i^* \varphi_i t_i$, where the $\varphi_i \in S_H(\mathscr{A})$, and t_i are invertible bounded operators with $\sum_{i=1}^n t_i^* t_i = 1$, then φ is unitary equivalent to each φ_i $(i = 1, \ldots, n)$. These three notions coincide for states, but are inequivalent starting with dimension $dimH \geq 2$. It is known that a pure map is a C^*-extreme point and that the C^*-extreme point is extreme provided that $dimH < \infty$. Many results indicates that C^*-extreme points have an interesting and relevant structure. Motivated by the restriction property of states (which means one-dimensional unital completely positive maps), we can ask whether the same holds for various types of extreme points in the generalized state space. For pure maps we immediately obtain the following counterexample:

Counterexample. Let $\mathscr{A}$ be an irreducible C^*-algebra acting on a separable Hilbert space H with $dimH \geq 2$. Then the embedding i of $\mathscr{A}$ into $B(H)$ is a pure map which is not pure on any maximal Abelian subalgebra of $\mathscr{A}$.

Proof. According to the extreme point theorem of Arveson [3], a completely positive map is pure if and only if its representation in the minimal Stinespring's decomposition is irreducible. So i is pure. Let $\mathscr{B}$ be an arbitrary maximal Abelian subalgebra of $\mathscr{A}$. Then $\mathscr{B}$ has a cyclic vector [e.g., 12, 13] and therefore $i|\mathscr{B}$ is equal to its representation in the minimal Stinespring decomposition. On the other hand, $\mathscr{B}$ cannot act irreducibly on H because its commutant is $\mathscr{B}$. Hence, $i|\mathscr{B}$ is not a pure map.

Let us remark that the map i in the proof of the previous counterexample was nevertheless multiplicative and thereby a C^*-extreme point on $\mathscr{B}$. In fact, using Theorem 1 and very recent results on structure of C^*-extreme points due to Farenick *et al.* [6, 7], we can prove that finite-dimensional C^*-extreme point enjoys the restriction property:

Theorem 2. Let φ be a C^*-extreme point of $S_H(\mathscr{A})$, where $dimH < \infty$, and let the representation π_φ corresponding to φ have σ-finite range. Then there is a maximal Abelian subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that $\varphi|\mathcal{B}$ is multiplicative on $\mathcal{B}$ and thereby a C^*-extreme point of $S_H(\mathcal{B})$.

For $dimH = 1$, Theorem 2 reduces to the above results on the restriction property of pure states on C^*-algebras.

2. EXTENSION THEOREM

In the concluding part of this paper we would like to consider the extension problem for vector measures on quantum structures. The classical Horn–Tarski theorem says that any (finitely additive) probability measure on a Boolean algebra extends to a probability measure on any larger Boolean algebra. Quantum measure-theoretic generalization of this result has been given in refs. 9 and 15. Recently we have succeeded in generalizing these results by (1) taking quotions of JBW algebras, (2) allowing more general embedding of substructures, and (3) considering Stone-algebra valued measures [10]. By a Stone algebra (boundedly complete vector lattice) we mean the space $C(X)$ of all continuous functions on a hyperstonean compact space X.

Theorem 3. Let M be a quotient of a JBW algebra W, where W does not contain type I_2 direct summand. Let μ be a positive S-valued measure on projection structure $P(M)$, where $(S, \leq, 1_S)$ is a boundedly complete vector lattice, such that $\mu(1_M) = 1_S$. Then μ extends to a positive S-valued measure on any larger generalized orthomodular lattice K with a unital set of states containing $P(M)$.

Let us note that that the set of all measures on projections of a JBW algebra with values in an infinite-dimensional Stone algebra does not form a compact space when endowed with pointwise-convergence topology. For that reason Theorem 3 cannot by proved by standard compactness arguments. A functional analytic component of this result is the fact that any unital positive map between order-unit vector space and Stone algebra can be extended to a positive unital map on a larger order-unit vector space [14]. We get from the category of orthomodular lattices to order-unit spaces by considering bounded affine functions on state spaces and employing Gleason's theorem and Theorem 1.

It was proved by Wright [17] that precisely Stone algebras are injective in the category of order-unit vector spaces and unital positive maps. This together with the previous discussion indicates that Theorem 3 cannot be improved by taking measures more general than Stone-algebra-valued.

Finally, we mention one consequence of Theorem 3. Let us take a complete Boolean algebra B with a separating set of completely additive measures which sits inside some orthomodular lattice K. Then the identity map i on B can be viewed as a unital positive measure with values in the Stone algebra $C(X)$ corresponding to B. By Theorem 3 we can extend i to a positive $C(X)$-valued measure E on K. Moreover, E is induced by a norm-one projection between corresponding order-unit spaces and can be therefore viewed as a lattice-theoretic counterpart of the conditional expectation in operator algebra theory.

ACKNOWLEDGMENTS

The author would like to express his gratitude to the Alexander von Humboldt Foundation for the support of the research in this paper. He would also like to thank to the Czech Technical University for supporting his research activity (grant no. J04/98/210000010).

REFERENCES

1. J. F. Aarnes and R. V. Kadison, Pure states and approximate identities, *Proc. Am. Math. Soc.* **21** (1969) 749–752.
2. C. A. Akemann, Approximate units and maximal Abelian C^*-subalgebras, *Pacific J. Math.* **33** (1970) 543–550.
3. W. Arveson, Subalgebras of C^*-algebras, *Acta Math.* **123**, 141–224.
4. B. A. Barnes, Pure states with the restriction property, *Proc. Am. Math. Soc.* **33** (1972) 491–494.
5. J. Bunce, Characters on singly generated C^*-algebras, *Proc. Am. Math. Soc.* **25** (1970) 297–303.
6. D. R. Farenick and P. B. Morenz, C^*-extreme points in the generalized state space of a C^*-algebra, *Trans. Am. Math. Soc.* **349** (1997) 1725–1748.
7. D. R. Farenick and H. Zhou, The structure of C^*-extreme points in space of completely positive linear maps on C^*-algebras, *Proc. Am. Math. Soc.* **126** (1998) 1467–1477.
8. H. Hanche-Olsen and E. Stormer, *Jordan Operator Algebras*, Pitman, Boston (1984).
9. J. Hamhalter, Gleason property and extensions of states on projection logics, *Bull. Lond. Math. Soc.* **26**, 367–372.
10. J. Hamhalter, Universal state space embeddability of Jordan–Banach algebras, *Proc. Am. Math. Soc.* **127** (1999) 131–137.
11. J. Hamhalter, Restricting pure states on JB algebras to maximal associative subalgebras, *Tatra Mountains Math. Publ.* **15** (1998) 165–171.
12. R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, Academic Press, New York (1986).
13. G. K. Pedersen, *C^*-Algebras and Their Automorphism Groups*, Academic Press, London (1979).
14. L. Nachbin, A theorem of Hahn–Banach type for linear transformations, *Trans. Am. Math. Soc.* **68** (1950) 265–247.
15. P. Pták, On extensions of states on logics, *Bull. Polish Acad. Sci. Math.* **33** (1985) 493–497.
16. G. A. Raggio, States and composite systems in W^*-algebraic quantum mechanics, Dissertation ETH, No. 6824, Zürich (1981).
17. J. D. M. Wright, An extension theorem and a dual proof of a theorem of Gleason, *J. Lond. Math. Soc.* **43** (1968) 699–702.